

UNIQUENESS FOR SOME CLASSES OF PARABOLIC PROBLEMS

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ABSTRACT. We prove some uniqueness results for weak solutions to some classes of parabolic Dirichlet problems.

1. INTRODUCTION

In the present paper we investigate the uniqueness of weak solutions to the following class of parabolic Dirichlet problems

$$(1.1) \quad \begin{cases} u_t - \operatorname{div} a(x, t, u, \nabla u) + H(x, t, \nabla u) + G(x, t, u) = f & \text{in } Q_T := \Omega \times (0, T) \\ u(x, t) = 0 & \text{in } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases}$$

where Ω is a bounded open subset of \mathbb{R}^N , $N \geq 2$, $p > 1$ and $T > 0$. We assume that $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$, $H : Q_T \times \mathbb{R}^N \rightarrow \mathbb{R}$ and $G : Q_T \times \mathbb{R} \rightarrow \mathbb{R}$ are Carathéodory functions satisfying the following structural conditions

$$(1.2) \quad a(x, t, s, \xi) \cdot \xi \geq \alpha_1 |\xi|^p,$$

$$(1.3) \quad |a(x, t, s, \xi)| \leq \beta_1 \left[|s|^{p-1} + |\xi|^{p-1} + a_1(x, t) \right],$$

$$(1.4) \quad (a(x, t, s, \xi) - a(x, t, s, \xi'))(\xi - \xi') > 0 \quad \text{for } \xi \neq \xi',$$

$$(1.5) \quad |H(x, t, \xi)| \leq b(x, t) |\xi|^\gamma$$

and

$$(1.6) \quad |G(x, t, s)| \leq c(x, t) |s|^\lambda$$

for a.e. $(x, t) \in Q_T$, $\forall s \in \mathbb{R}, \forall \xi, \xi' \in \mathbb{R}^N$, where α_1 and β_1 are positive constants, $\gamma = p - \frac{N+p}{N+2}$, $\lambda = p \frac{N+2}{N+1}$, $a_1 \in L^{p'}(Q_T)$, $b \in L^r(Q_T)$ and $c \in L^\rho(Q_T)$ with $r = N+2$ and $\rho = \frac{N+p}{N}$. Moreover $f = f_0 - \operatorname{div} F$ with

$$(1.7) \quad f_0 \in L^{\left(\frac{p(N+2)}{N}\right)'}(Q_T), F \in \left(L^{p'}(Q_T)\right)^N \quad \text{and } u_0 \in L^2(\Omega).$$

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We recall that a weak solution¹ to Problem (1.1) is a measurable function belonging to $C(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ such that $\forall t \in (0, T]$

$$(1.8) \quad \int_{\Omega} uv(x, t) dx + \iint_{Q_t} [-uv_t + a(u, \nabla u) \nabla v + H(\nabla u) v + G(u) v] dx d\tau \\ = \int_{\Omega} u_0 v(x, 0) dx + \iint_{Q_t} (f_0 v + F \nabla v) dx d\tau,$$

for each $v \in W_0^{1,2}(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$, where $Q_t := \Omega \times (0, t)$.

If the function a does not depend on u and $H \equiv G \equiv 0$, we know there exists a unique weak solution (see *e.g.* [22]). Here we prove the uniqueness of weak solutions to Problem (1.1) when $H \equiv 0$ and $G \neq 0$ assuming that operator $-\operatorname{div} a(x, t, u, \nabla u)$ is strongly monotone and the functions a and G are Lipschitz continuous with respect to u (that is usual as far as uniqueness result concern). When $p \geq 2$ we assume the principle part is not degenerate, *i.e.* in the model case $-\operatorname{div} a(x, t, u, \nabla u) = -\operatorname{div}(a_0(x, t, u)(\varepsilon + |\nabla u|^{p-1} \nabla u))$ with $\varepsilon > 0$. In this case we can relax hypothesis on the function a assuming only a locally Lipschitz continuity with respect to u (see Section 2 for details). To our knowledge in literature there are not any existence results for weak solutions to Problem (1.1) when $H \equiv 0$ and $G \neq 0$, then we will give some details (see Proposition 3.1) for convenience of the reader.

When $H \neq 0$ and $G \equiv 0$, the existence of a weak solution to problem (1.1) is investigated in [23]. If a does not depend on u and under Lipschitz continuity on the lower order term H , we prove that such a solution is unique.

Our proof of uniqueness adapts the idea of [1] (used also in [18] for an anisotropic elliptic operator) to the evolution case: the main tool is the embedding in the parabolic equation framework. In the case $H \equiv 0$ and $G \neq 0$ the method is improved using Gronwall's Lemma. Our technique works also in proving some comparison principles.

There is an extensive literature about uniqueness of solution for elliptic equations. We just mention some of these papers: *e.g.* [2], [3], [4], [7], [8], [9], [12], [13], [14] and [19]. For the evolution case some uniqueness results can be found for example in [6], [20], [21] and [24] in the framework of weak solutions. When datum f is only integrable, uniqueness of renormalized and entropy solutions is proved for example in [5], [10], [11], [17] and [25].

2. STATEMENTS OF RESULTS

First we study the case $H \equiv 0$, that is we consider the following class of nonlinear parabolic homogeneous Dirichlet problems

$$(2.1) \quad \begin{cases} u_t - \operatorname{div} a(x, t, u, \nabla u) + G(x, t, u) = f & \text{in } Q_T \\ u(x, t) = 0 & \text{in } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases}$$

when (1.2)-(1.4) and (1.6)-(1.7) hold, function a also satisfies the following strong monotony condition

$$(2.2) \quad (a(x, t, s, \xi) - a(x, t, s, \xi'))(\xi - \xi') \geq \alpha(\varepsilon + |\xi| + |\xi'|)^{p-2} |\xi - \xi'|^2$$

¹We refer to [15] for definitions of involved function spaces and parabolic framework.

and the following Lipschitz continuity condition

$$(2.3) \quad |a(x, t, s, \xi) - a(x, t, s', \xi)| \leq \beta \left[\phi + |\xi|^{p-1} + (|s| + |s'|)^\theta \right] |s - s'|$$

and function G also satisfies the following Lipschitz continuity condition

$$(2.4) \quad |G(x, t, s, \cdot) - G(x, t, s', \cdot)| \leq \varrho |s - s'|$$

for some $\theta \geq 0$ with $\alpha > 0$, $\varepsilon \geq 0$, $\beta, \varrho > 0$ and $\phi \geq 0$.

We investigate separately the case $1 < p < 2$ and $p \geq 2$.

Theorem 2.1. *Let us assume $1 < p < 2$, (1.2)-(1.3), (1.6)-(1.7), (2.2) with $\varepsilon = 0$, (2.3) with $\theta = 0$ and (2.4) hold. Then there exists a unique weak solution to Problem (2.1).*

Theorem 2.2. *Let us assume $p \geq 2$, (1.2)-(1.3), (1.6)-(1.7), (2.2) with $\varepsilon > 0$, (2.3) with $0 \leq \theta \leq \frac{p(N+2)}{2N}$ and (2.4) hold. Then there exists a unique weak solution to Problem (2.1).*

Remark 2.1. *Theorem 2.1 holds if we replace $|s - s'|$ in (2.3) and (2.4) by $\omega(|s - s'|)$, where $\omega : [0, +\infty[\rightarrow [0, +\infty[$ is such that $\omega(s) \leq s$ for $0 \leq s \leq \kappa$ for some $\kappa > 0$. Analogues generalization holds for Theorem 2.2.*

Moreover in order to prove uniqueness results for problems with lower order term $H(x, t, \nabla u)$ we suppose function a does not depend on u and $G \equiv 0$. More precisely we study the following class of nonlinear parabolic homogeneous Dirichlet problems

$$(2.5) \quad \begin{cases} u_t - \operatorname{div} a(x, t, \nabla u) + H(x, t, \nabla u) = f & \text{in } Q_T \\ u(x, t) = 0 & \text{in } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases}$$

when (1.2)-(1.5) and (1.7) hold. We know (see [23]) there exists at least a weak solution to Problem (2.5). As usual we assume the following locally Lipschitz condition on H

$$(2.6) \quad |H(x, t, \xi) - H(x, t, \xi')| \leq h(x, t) (\eta + |\xi| + |\xi'|)^\sigma |\xi - \xi'|$$

with $\sigma \in \mathbb{R}$, $\eta \geq 0$ and h a suitable function.

We investigate separately the case $p < 2$ and $p \geq 2$.

Theorem 2.3. *Let us assume $\frac{2N}{N+2} \leq p < 2$, (1.2)-(1.3), (1.5), (1.7), (2.2) with $\varepsilon = 0$ and (2.6) with $h \in L^\infty(Q_T)$, $\eta > 0$ and $\sigma \leq \frac{p-2}{2}$ hold. Then there exists a unique weak solution to Problem (2.5).*

Theorem 2.4. *Let us assume (1.2)-(1.3), (1.5), (1.7), (2.2) with $\varepsilon > 0$ and (2.6) with $h \in L^r(Q_T)$ for $r \geq N+2$, $\eta = 0$ and $0 \leq \sigma \leq p \left(\frac{1}{N+2} - \frac{1}{r} \right) + \frac{p-2}{2}$ hold. Then for $2 \leq p \leq \frac{2r(N+2)}{r(N+2)+2(N+2)-2r}$ there exists a unique weak solution to Problem (2.5).*

Remark 2.2. *If $p > \frac{2r(N+2)}{r(N+2)+2(N+2)-2r}$, then Theorem 2.4 holds with $0 \leq \sigma \leq \frac{p}{N+2} - \frac{p}{r}$ and $\frac{p-2}{2} \leq \sigma \leq \frac{p}{N+2} - \frac{p}{r} + \frac{p-2}{2}$.*

Remark 2.3. *If in Problem (2.5) we add extra term $G(x, t, u)$ that is an increasing function in the variable u , the uniqueness of weak solutions can be proved under hypothesis of Theorems 2.3 and 2.4.*

The arguments used in the proofs of previous theorems allows us to obtain also some comparison principles.

Corollary 2.1. (Comparison principle) *In the hypothesis of Theorems 2.1 and 2.2, let us assume u and v are two solutions to Problem (2.1) such that $u(x, 0) \leq v(x, 0)$ a.e. in Ω . Then $u_1 \leq u_2$ a.e. in Q_T .*

Corollary 2.2. (Comparison principle) *In the hypothesis of Theorems 2.3 and 2.4, let us assume u and v are two solutions to Problem (2.5) such that $u(x, 0) \leq v(x, 0)$ a.e. in Ω . Then $u_1 \leq u_2$ a.e. in Q_T .*

3. OPERATORS WITH A ZERO ORDER TERM

In this section we study Problem (2.1) when (1.2)-(1.4) and (1.6)-(1.7) hold.

3.1. Some preliminary results. In order to prove theorems of the previous section we need to recall the following embedding in the parabolic framework.

Lemma 3.1. (see Proposition 3.1 of [15]) *Let $u \in L^\infty(0, T; L^\rho(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ with $p \geq 1$ and $\rho \geq 1$. Then $u \in L^q(Q_T)$ with $q = p \frac{N+\rho}{N}$ and there exists a constant C_p that depends on N and p such that*

$$(3.1) \quad \|u\|_{L^q(Q_T)} \leq C_p \left(\sup_{0 < t < T} \|u(\cdot, t)\|_{L^\rho(\Omega)} + \|\nabla u\|_{L^p(Q_T)} \right).$$

Moreover it results

$$(3.2) \quad \iint_{Q_T} |u|^q \leq C_p^q \left(\sup_{0 < t < T} \int_{\Omega} |u|^\rho \right)^{\frac{p}{N}} \iint_{Q_T} |\nabla u|^p.$$

Moreover in the proof of uniqueness result for Problem (2.1) we need the following version of Gronwall lemma.

Lemma 3.2. *Let $T > 0$ and let a, d be non-decreasing functions belonging to $L^1_{loc}(\mathbb{R}_+)$, $b \in L^\infty_{loc}(\mathbb{R}_+)$ and $z \in L^1_{loc}(\mathbb{R}_+)$ such that*

$$z(t) \leq a(t) + d(t) \int_0^t b(s) z(s) ds \quad \text{for a.e. } t \in [0, T],$$

then

$$(3.3) \quad z(t) \leq a(t) \left[1 + d(t) \int_0^t b(s) ds \exp \left(d(t) \int_0^t b(s) ds \right) \right] \quad \text{for a.e. } t \in [0, T].$$

3.2. Existence of a weak solution. To our knowledge in literature there are not existence results for weak solutions to Problem (2.1).

Proposition 3.1. *Under assumptions (1.2)-(1.4) and (1.6)-(1.7) there exists at least a weak solution $u \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$ to Problem (2.1).*

The proof is standard but for convenience of reader we will give here some steps. We observe that the coercivity of the operator is guaranteed only if the norm $\|c\|_{L^\rho(Q_T)}$ is small enough. Then as usual we consider the approximate problems

$$(3.4) \quad \begin{cases} (u_n)_t + L_n u_n = f_n - \operatorname{div} F & \text{in } Q_T \\ u_n(x, t) = 0 & \text{in } \partial\Omega \times (0, T) \\ u_n(x, 0) = u_0(x) & \text{on } \Omega, \end{cases}$$

where $L_n u = -\operatorname{div} a(x, t, u, \nabla u) + G_n(x, t, u)$, $G_n(x, t, s) = T_n(G(x, t, s))$, T_n is the truncation at level $\pm n$, defined by

$$(3.5) \quad T_n(s) = \max \{-n, \min \{n, s\}\}$$

and $\{f_n\}_{n \in \mathbb{N}} \subset L^{p'}(Q_T)$ such that with $f_n \rightarrow f_0$ strongly in $L^{q'}(Q_T)$ with $q = \frac{p(N+2)}{N}$. The operator L_n is pseudomonotone and coercive, then (see e.g. [22]) there exists a weak solution $u_n \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$. The following a priori estimate of u_n holds.

Lemma 3.3. *Assume that (1.2)-(1.4) and (1.6)-(1.7) hold. If u_n is a weak solution to Problem (3.4), then there exists a constant C_0 (depending on the data that appear in the structure conditions but not on n) such that*

$$(3.6) \quad \sup_{0 < t < T} \|u_n\|_{L^2(\Omega)} + \|\nabla u_n\|_{L^p(Q_T)} \leq C_0.$$

Proof. Using u_n as test function in Problem (3.4), under assumptions (1.2) and (1.6) we obtain for $t \in (0, T)$

$$(3.7) \quad \frac{1}{2} \int_{\Omega} u_n^2(t) dx + \alpha_1 \iint_{Q_t} |\nabla u_n|^p dx d\tau \leq \iint_{Q_t} c |u_n|^{\lambda+1} dx d\tau + \frac{1}{2} \int_{\Omega} u_0^2 dx + \iint_{Q_t} (f_n u_n + F \nabla u_n) dx d\tau.$$

Using Hölder inequality, (3.1) and Young inequality we have

$$(3.8) \quad \iint_{Q_t} (f_0 u_n + F \nabla u_n) dx d\tau \leq \frac{\alpha_1}{p} \|\nabla u_n\|_{L^p(Q_t)}^p + \kappa_1 \|F\|_{L^{p'}(Q_t)} + \kappa_2 \|f_n\|_{L^{q'}(Q_t)}^{\frac{p(N+2)}{p(N+1)-N}} + \kappa_3 \left[\int_{\Omega} u_n^2(t) dx + \|\nabla u_n\|_{L^p(Q_t)}^p \right]$$

for some positive constant κ_1, κ_2 and κ_3 with $\kappa_3 < \min \left\{ \frac{1}{2}, \frac{\alpha_1}{p'} \right\}$. Using (3.1) and Young inequality, we get

$$(3.9) \quad \iint_{Q_t} c(x, \tau) |u_n|^{\lambda+1} dx d\tau \leq \|c\|_{L^{\rho}(Q_t)} \left(\sup_{0 < \tau < t} \int_{\Omega} u_n^2(\tau) dx \right)^{\frac{p}{N\rho'}} \left(\iint_{Q_t} |\nabla u_n|^p dx d\tau \right)^{\frac{1}{\rho'}} \leq \kappa_4 \|c\|_{L^{\rho}(Q_t)} \left[\sup_{0 < \tau < t} \int_{\Omega} u_n^2(\tau) dx + \iint_{Q_t} |\nabla u_n|^p dx d\tau \right]$$

for some positive constant κ_4 . Using (3.8) and (3.9) in (3.7) and taking the supremum on $(0, t_1]$ for some $t_1 \leq T$ such that $\|c\|_{L^{\rho}(Q_{t_1})}$ is small enough, we obtain

$$(3.10) \quad \sup_{t \in (0, t_1]} \int_{\Omega} u_n^2(t) dx + \iint_{Q_{t_1}} |\nabla u_n|^p dx dt \leq \kappa_5 \left[\int_{\Omega} u_0^2 dx + \|F\|_{L^{p'}(Q_{t_1})} + 1 \right],$$

for some positive constant κ_5 , since $\{f_n\}_{n \in \mathbb{N}}$ is bounded in $L^{q'}(Q_T)$. In order to avoid the assumption on smallness of the norm $\|c\|_{L^{\rho}(Q_T)}$ we split (see also [23] and [16]) the interval $[0, T]$ in M small subinterval (t_i, t_{i+1}) for $i = 0, \dots, M-1$ in such a way $\|c\|_{L^{\rho}(\Omega \times (t_i, t_{i+1}))}$ is small enough. We are able to derive an estimate like (3.10) for small cylinder $\Omega \times (t_i, t_{i+1})$. Finally taking the sum of different iterations, (3.6) holds for the inter cylinder Q_T . \square

Proof of Proposition 3.1. By Lemma 3.3 it follows that $\{u_n\}_{n \in \mathbb{N}}$ is bounded sequence of $L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$. Then it is possible to proceed as in the proof of Theorem 2.2 of [23] to pass to the limit in (3.4) and to conclude the existence of at least a weak solution to Problem (2.1).

We explicitly write only the computation about the boundness of $G_n(x, t, u_n)$ in $L^{q'}(Q_T)$. Ended by (1.6), Hölder inequality and (3.1) we have

$$\begin{aligned} \iint_{Q_T} |G_n(x, t, u_n)|^{q'} dx dt &\leq \iint_{Q_T} c(x, t)^{q'} |u_n|^{\lambda q'} dx dt \leq \|c\|_{L^\rho(Q_t)}^{1 - \frac{\lambda q'}{q}} \left(\iint_{Q_T} u_n^q \right)^{\frac{\lambda q'}{q}} \\ &\leq \|c\|_{L^\rho(Q_t)}^{1 - \frac{\lambda q}{q'}} \left[C_p \left(\sup_{0 < t < T} \|u_n(\cdot, t)\|_{L^2(\Omega)} + \|\nabla u_n\|_{L^p(Q_T)} \right) \right]^{\lambda q'}. \end{aligned}$$

By Lemma 3.3 the norm of $G_n(x, t, u_n)$ in $L^{q'}(Q_T)$ is bounded by a constant depending on the data that appear in the structure conditions but not on n . \square

3.3. Uniqueness of weak solutions. In this subsection we prove *ab absurdo* the uniqueness of weak solutions to Problem (2.1).

Proof of uniqueness in the hypothesis of Theorem 2.1. We argue by contradiction. Let us assume that Problem (2.1) admits two different solutions u and v and $D = \{(x, t) \in Q_T : w > 0\}$ has positive measure, where $w = u - v$. Using $\varphi = \frac{T_k(w^+)}{k}$ for $k \in \left[0, \sup_D w^+\right]$ as test function in the difference of the equations (where $T_k(\cdot)$ is defined in (3.5)), we obtain for $t \in (0, T)$

$$\int_{\Omega} w \varphi + \iint_{Q_t} \{-w \varphi_t + [a(x, t, u, \nabla u) - a(x, t, v, \nabla v)] \nabla \varphi + [c(x, t, u) - c(x, t, v)] \varphi\} = 0.$$

Let us denote $\Psi_k(s) = \int_0^s T_k(\sigma) d\sigma$. We have that

$$(3.11) \quad \int_{\Omega} w \varphi - \iint_{Q_t} w \varphi_t = \frac{1}{k} \int_{\Omega} \Psi_k(w^+(t))$$

for $k > 0$. By (2.2), (2.3), (2.4) (3.11) we get

$$\begin{aligned} (3.12) \quad &\frac{1}{k^2} \int_{\Omega} \Psi_k(w^+(t)) + \alpha \iint_{Q_t \cap D_k} \frac{|\nabla \varphi|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \\ &\leq \beta \iint_{Q_t \cap D_k} (\phi + |\nabla v|^{p-1}) |\nabla \varphi| + \frac{\rho}{k} \iint_{Q_t} |w| \varphi, \end{aligned}$$

where $D_k = \{(x, t) \in D : w^+ < k\}$. Using Young inequality with some $\delta > 0$ it follows

$$\begin{aligned} (3.13) \quad &\iint_{Q_t \cap D_k} (\phi + |\nabla v|^{p-1}) |\nabla \varphi| \leq \frac{\delta(\phi + 1)}{2} \iint_{Q_t \cap D_k} \frac{|\nabla \varphi|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \\ &\quad + \frac{\phi}{4\delta} \iint_{Q_t \cap D_k} (|\nabla u| + |\nabla v|)^{2-p} + \frac{1}{4\delta} \iint_{Q_t \cap D_k} (|\nabla u| + |\nabla v|)^p. \end{aligned}$$

Choosing δ small enough, using (3.13) and Young inequality in (4.6) and noticing that $\Upsilon(s) = 2\Psi_k(s) - sT_k(s) \geq 0$ for $s \geq 0$ (check that $\Upsilon(s) = 0$ for $0 \leq s \leq k$ and

$\Upsilon'(s) \geq 0$), we obtain

$$(3.14) \quad \begin{aligned} & \frac{1}{k^2} \int_{\Omega} \Psi_k(w^+(t)) + c_1 \iint_{Q_t \cap D_k} \frac{|\nabla \varphi|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \\ & \leq c_2 \left(|D_k \cap Q_t| + \iint_{Q_t \cap D_k} (|\nabla u| + |\nabla v|)^p + \frac{1}{k^2} \iint_{Q_t} \Psi_k(w^+) \right) \end{aligned}$$

for some positive constants c_1, c_2 independent on k .

Using Gronwall inequality (3.3) and taking the supremum on $t \in (0, T)$, we get

$$(3.15) \quad \frac{1}{k^2} \sup_{t \in (0, T)} \int_{\Omega} \Psi_k(w^+(t)) \leq c_2 (1 + Te^T) \left(|D_k| + \iint_{D_k} (|\nabla u| + |\nabla v|)^p \right).$$

It is easy to check that

$$(3.16) \quad \zeta_1(k) := \left[|D_k| + \iint_{D_k} (|\nabla u| + |\nabla v|)^p \right] \rightarrow 0$$

when k goes to zero, then

$$(3.17) \quad \lim_{k \rightarrow 0} \frac{1}{k^2} \sup_{t \in (0, T)} \int_{\Omega} \Psi_k(w^+(t)) = 0.$$

Recalling that $\frac{1}{2} |T_k(s)|^2 \leq \Psi_k(s)$, we have

$$(3.18) \quad \lim_{k \rightarrow 0} \sup_{t \in (0, T)} \int_{\Omega} |\varphi|^2 = 0.$$

Coming back to (3.14), taking the supremum and using (3.17) and (3.16), we obtain

$$(3.19) \quad \lim_{k \rightarrow 0} \iint_{D_k} \frac{|\nabla \varphi|^2}{(|\nabla u| + |\nabla v|)^{2-p}} = 0.$$

Moreover inequality (3.2) and Hölder inequality imply

$$\begin{aligned} C_1^{-\frac{N+2}{N}} |D \setminus D_k| & \leq C_1^{-\frac{N+2}{N}} \|\varphi\|_{L^{\frac{N+2}{N}}(D)}^{\frac{N+2}{N}} \leq \left(\sup_{t \in (0, T)} \int_{\Omega} |\varphi|^2 \right)^{\frac{1}{N}} \iint_{D_k} |\nabla \varphi| = \\ & \leq \left(\sup_{t \in (0, T)} \int_{\Omega} |\varphi|^2 \right)^{\frac{1}{N}} \left(\iint_{D_k} \frac{|\nabla \varphi|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \right)^{\frac{1}{2}} \left(\iint_{D_k} (|\nabla u| + |\nabla v|)^{2-p} \right)^{\frac{1}{2}}. \end{aligned}$$

By (3.16), (3.18) and (3.19) it follows

$$(3.20) \quad |D| = \lim_{k \rightarrow 0} |D \setminus D_k| = 0.$$

To complete the proof it suffices to replace u and v . □

Proof of uniqueness in the hypothesis of Theorem 2.2. Arguing as in the previous proof and taking into account the following extra term

$$\iint_{Q_t \cap D_k} (|u| + |v|)^{\theta} |\nabla \varphi|,$$

we obtain

$$(3.21) \quad \begin{aligned} & \frac{1}{k^2} \int_{\Omega} \Psi_k(w^+(t)) + \alpha \iint_{Q_t \cap D_k} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla \varphi|^2 \\ & \leq \beta \iint_{Q_t \cap D_k} \left[\phi + |\nabla v|^{p-1} + (|u| + |v|)^{\theta} \right] |\nabla \varphi| + \frac{2\rho}{k^2} \iint_{Q_t} \Psi_k(w^+). \end{aligned}$$

Using Young inequality with some $\delta > 0$, we have the analogue of (3.13), i.e.

$$(3.22) \quad \begin{aligned} & \iint_{Q_t \cap D_k} \left[\phi + |\nabla v|^{p-1} + (|u| + |v|)^{\theta} \right] |\nabla \varphi| \\ & \leq \frac{\delta}{2} \left[\frac{(\phi + 1)}{\varepsilon^{p-2}} + 1 \right] \iint_{Q_t \cap D_k} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla \varphi|^2 \\ & \quad + \frac{\phi}{4\delta} |Q_t \cap D_k| + \frac{1}{4\delta} \iint_{Q_t \cap D_k} |\nabla v|^p + \frac{1}{4\delta} \iint_{Q_t \cap D_k} (|u| + |v|)^{2\theta}. \end{aligned}$$

Choosing δ small enough in (3.22), inequality (3.21) gets

$$(3.23) \quad \begin{aligned} & \frac{1}{k^2} \int_{\Omega} \Psi_k(w^+(t)) + c_1 \iint_{Q_t \cap D_k} (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} |\nabla \varphi|^2 \\ & \leq c_2 \left[|D_k| + \iint_{Q_t \cap D_k} |\nabla v|^p + \iint_{Q_t \cap D_k} (|u| + |v|)^{2\theta} + \frac{1}{k^2} \iint_{Q_t} \Psi_k(w^+) \right]. \end{aligned}$$

for some positive constants c_1, c_2 independent on k . We remark that since $2\theta \leq \frac{p(N+2)}{N}$

$$(3.24) \quad \zeta_2(k) := |D_k| + \iint_{D_k} |\nabla v|^p + \iint_{D_k} (|u| + |v|)^{2\theta} \rightarrow 0$$

when k goes to zero. Arguing as in the next proof we obtain (3.18) and

$$(3.25) \quad \lim_{k \rightarrow 0} \iint_{D_k} |\nabla \varphi|^2 = 0.$$

Moreover inequality (3.1) and Young inequality imply

$$(3.26) \quad C_1^{-1} |D \setminus D_k|^{\frac{N}{N+2}} \leq \sup_{t \in (0, T)} \left(\int_{\Omega} |\varphi|^2 \right)^{\frac{1}{2}} + \frac{1}{2} \iint_{D_k} |\nabla \varphi|^2 + \frac{1}{2} |D_k|.$$

By (3.26), (3.16), (3.18) and (3.25) it follows (3.20). Then the assert holds changing the role of u and v . \square

Remark 3.1. If $G(x, t, u) = c(x)u$ the previous proofs follow easier multiplying equation by $\exp(tc(x))$ and using as test function $\varphi = \frac{T_k((u-v)^+ \exp(tc(x)))}{k}$.

Proof of Corollary 2.1. We can argue as in the previous proofs of uniqueness putting $w = (u - v)^+$. \square

4. OPERATORS WITH A FIRST ORDER TERM

We known that there exists at least a weak solution to Problem (2.5). Here we have to prove only the uniqueness.

Proof of Theorem 2.3. Let us suppose that u and v are two weak solutions to Problem (2.5) belonging to $L^\infty(0, T, L^2(\Omega)) \cap L^p(0, T, W_0^{1,p}(\Omega))$ such that $w = u - v > 0$ in a subset $D \subset Q_T$ with $|D| > 0$. Let us denote

$$w_k = \begin{cases} w^+ - k & \text{if } w^+ > k \\ 0 & \text{otherwise} \end{cases}$$

for $k \in \left(0, \sup_D w^+\right)$. We use w_k as test function in the difference of the equation:

$$(4.1) \quad \begin{aligned} \int_{\Omega} w w_k + \iint_{Q_t} [-w(w_k)_t + (a(x, t, u, \nabla u) - a(x, t, v, \nabla v)) \nabla w_k] \\ \leq \iint_{Q_t} |H(x, t, \nabla v) - H(x, t, \nabla u)| w_k \end{aligned}$$

for $t \in (0, T)$. We observe that

$$(4.2) \quad \int_{\Omega} w w_k - \iint_{Q_t} w(w_k)_t = \frac{1}{2} \int_{\Omega} w_k^2(t).$$

Using (4.2), (2.2) and (2.6) with $h \in L^\infty(Q_T)$, $\eta > 0$ and $\sigma \leq \frac{p-2}{2}$, inequality (4.1) becomes

$$\frac{1}{2} \int_{\Omega} w_k^2(t) + \alpha \iint_{Q_t} \frac{|\nabla w_k|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \leq h \iint_{Q_t} \frac{|\nabla w_k| w_k}{(\eta + |\nabla u| + |\nabla v|)^{-\sigma}}.$$

Taking the supremum on $t \in (0, T)$ we obtain

$$(4.3) \quad \frac{1}{2} \sup_{0 < t < T} \int_{\Omega} w_k^2 + \alpha \iint_{E_k} \frac{|\nabla w_k|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \leq h \iint_{E_k} \frac{|\nabla w_k| w_k}{(\eta + |\nabla u| + |\nabla v|)^{-\sigma}},$$

where $E_k = \left\{ (x, t) \in Q_T : k < w^+ < \sup_D w \right\}$. Since $\sigma \leq \frac{p-2}{2}$, Young inequality gets

$$(4.4) \quad \iint_{E_k} \frac{|\nabla w_k| w_k}{(\eta + |\nabla u| + |\nabla v|)^{\sigma}} \leq \frac{\delta}{2} \iint_{E_k} \frac{|\nabla w_k|^2}{(|\nabla u| + |\nabla v|)^{2-p}} + \frac{1}{4\delta\eta^{-2\sigma+p-2}} \iint_{E_k} |w_k|^2$$

for $\delta > 0$. Putting (4.4) in (4.3) and choosing δ small enough we have

$$(4.5) \quad \sup_{0 < t < T} \int_{\Omega} w_k^2 + \iint_{E_k} \frac{|\nabla w_k|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \leq c \iint_{E_k} |w_k|^2,$$

where c is a positive constant independent on k . Using (3.2), Hölder and Young inequalities and (4.5) we have

$$\begin{aligned}
\iint_{E_k} |w_k|^2 &\leq C_{\frac{2N}{N+2}}^2 \left[\sup_{0 < t < T} \int_{\Omega} |w_k|^2 \right]^{\frac{2}{N+2}} \iint_{E_k} |\nabla w_k|^{\frac{2N}{N+2}} \\
&\leq C_{\frac{2N}{N+2}}^2 \left[\sup_{0 < t < T} \int_{\Omega} |w_k|^2 \right]^{\frac{2}{N+2}} \left(\iint_{E_k} \frac{|\nabla w_k|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \right)^{\frac{N}{N+2}} \\
&\quad \times \left(\iint_{E_k} (|\nabla u| + |\nabla v|)^{(2-p)\frac{N}{2}} \right)^{\frac{2}{N+2}} \\
&\leq C_{\frac{2N}{N+2}}^2 \left[\frac{2}{N+2} \sup_{0 < t < T} \int_{\Omega} |w_k|^2 + \frac{N}{N+2} \iint_{E_k} \frac{|\nabla w_k|^2}{(|\nabla u| + |\nabla v|)^{2-p}} \right] \\
&\quad \times \left(\iint_{E_k} (|\nabla u| + |\nabla v|)^{(2-p)\frac{N}{2}} \right)^{\frac{2}{N+2}} \\
&\leq c \frac{C_{\frac{2N}{N+2}}^2 N}{N+2} \left(\iint_{E_k} |w_k|^2 \right) \left(\iint_{E_k} (|\nabla u| + |\nabla v|)^{(2-p)\frac{N}{2}} \right)^{\frac{2}{N+2}},
\end{aligned}$$

where $C_{\frac{2N}{N+2}}$ is the constant in (3.1). It easily follows that

$$1 \leq c \frac{C_{\frac{2N}{N+2}}^2 N}{N+2} \left(\iint_{E_k} (|\nabla u| + |\nabla v|)^{(2-p)\frac{N}{2}} \right)^{\frac{2}{N+2}}.$$

Since $p \geq \frac{2N}{N+2}$, the right-hand side goes to zero when k goes to $\sup_D w^+$, which is impossible. To complete the proof we have to change the role of u and v . \square

Proof of Theorem 2.4. We argue as in the proof of Theorem 2.3, obtaining

$$\begin{aligned}
(4.6) \quad &\frac{1}{2} \sup_{0 < t < T} \int_{\Omega} |w|^2 + \alpha \iint_{E_k} |\nabla w_k|^2 (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} \\
&\leq \iint_{E_k} h (|\nabla u| + |\nabla v|)^{\sigma} |\nabla w_k| w_k.
\end{aligned}$$

Let us suppose $\sigma \geq \frac{p-2}{2}$. Using Hölder inequality and inequality (3.1) we have

$$\begin{aligned}
(4.7) \quad &\iint_{E_k} h (|\nabla u| + |\nabla v|)^{\sigma} |\nabla w_k| w_k \\
&\leq \left(\iint_{E_k} \left(h (|\nabla u| + |\nabla v|)^{\sigma - \frac{p-2}{2}} \right)^{N+2} \right)^{\frac{1}{N+2}} \\
&\quad \times \left(\iint_{E_k} |\nabla w_k|^2 (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} \right)^{\frac{1}{2}} \left(\iint_{E_k} |w_k|^{\frac{2(N+2)}{N}} \right)^{\frac{N}{2(N+2)}} \\
&\leq C_2 \left(\iint_{E_k} \left(h (|\nabla u| + |\nabla v|)^{\sigma - \frac{p-2}{2}} \right)^{N+2} \right)^{\frac{1}{N+2}} \left(\iint_{E_k} |\nabla w_k|^2 (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} \right)^{\frac{1}{2}} \\
&\quad \times \left[\sup_{0 < t < T} \left(\int_{\Omega} |w_k|^2 \right)^{\frac{1}{2}} + \left(\iint_{E_k} |\nabla w_k|^2 \right)^{\frac{1}{2}} \right],
\end{aligned}$$

where C_2 is the constant in (3.1). Putting (4.7) in (4.6), by Young inequality and some easy computation we obtain

$$\begin{aligned} \min \left\{ \frac{1}{2}, \alpha \varepsilon^{p-2} \right\} & \left[\sup_{0 < t < T} \int_{\Omega} |w|^2 + \iint_{E_k} |\nabla w_k|^2 \right] \\ & \leq \sqrt{2} C_2 \max \left\{ 1, \frac{1}{\varepsilon^{\frac{p-2}{2}}} \right\} \left(\iint_{E_k} \left(h(|\nabla u| + |\nabla v|)^{\sigma - \frac{p-2}{2}} \right)^{N+2} \right)^{\frac{1}{N+2}} \times \\ & \left[\sup_{0 < t < T} \int_{\Omega} |w_k|^2 + \iint_{E_k} |\nabla w_k|^2 (\varepsilon + |\nabla u| + |\nabla v|)^{p-2} \right], \end{aligned}$$

i.e.

$$(4.8) \quad 1 \leq \frac{\sqrt{2} C_2 \max \left\{ 1, \frac{1}{\varepsilon^{\frac{p-2}{2}}} \right\}}{\min \left\{ \frac{1}{2}, \alpha \right\}} \left(\iint_{E_k} \left(h(|\nabla u| + |\nabla v|)^{\sigma - \frac{p-2}{2}} \right)^{N+2} \right)^{\frac{1}{N+2}}.$$

Since $\frac{N+2}{r} + \frac{(\sigma - \frac{p-2}{2})(N+2)}{p} \leq 1$, the right-hand side in (4.8) goes to zero when k goes to $\sup_D w$, which is impossible.

Conversely if $0 \leq \sigma < \frac{p-2}{2}$ as before we have

$$\begin{aligned} (4.9) \quad & \iint_{E_k} (|\nabla u| + |\nabla v|)^{\sigma} |\nabla w_k| w_k \\ & \leq \left(\iint_{E_k} (h(|\nabla u| + |\nabla v|)^{\sigma})^{N+2} \right)^{\frac{1}{N+2}} \left(\iint_{E_k} |\nabla w_k|^2 \right)^{\frac{1}{2}} \left(\iint_{E_k} |w_k|^{\frac{2(N+2)}{N}} \right)^{\frac{N}{2(N+2)}} \\ & \leq C_2 \left(\iint_{E_k} (h(|\nabla u| + |\nabla v|)^{\sigma})^{N+2} \right)^{\frac{1}{N+2}} \left(\iint_{E_k} |\nabla w_k|^2 \right)^{\frac{1}{2}} \\ & \times \left[\sup_{0 < t < T} \left(\int_{\Omega} |w_k|^2 \right)^{\frac{1}{2}} + \left(\iint_{E_k} |\nabla w_k|^2 \right)^{\frac{1}{2}} \right]. \end{aligned}$$

Then putting (4.9) in (4.6) by Young inequality and some easy computation we have

$$1 \leq \frac{\sqrt{2} C_2}{\min \left\{ \frac{1}{2}, \alpha \varepsilon^{p-2} \right\}} \left(\iint_{E_k} (h(|\nabla u| + |\nabla v|)^{\sigma})^{N+2} \right)^{\frac{2}{N+2}}.$$

Since $\frac{N+2}{r} + \frac{\sigma(N+2)}{p} \leq 1$, the contradiction follows again. Changing the role of u and v , we complete the proof. \square

Proof of Corollary 2.2. We can argue as in the proof of Theorems 2.3 and 2.4, putting $w = (u - v)^+$. \square

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